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Crack Detection, Localization and Estimation of the Depth in a Turbo Rotor

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The goal of this paper is to describe an advanced method of a crack detection: a new way to localize position and to estimate depth of a crack on rotating shaft. As a first step, the shaft is physically modelled with a finite element method and the dynamic mathematical model is derived using the Hamilton principle; thus, the system is represented by various subsystems. The equations of motion of the shaft with a crack are established by adapting the local stiffness change through breathing and gaping from the crack to an undamaged shaft. This is the reference system for the given system. Based on a model for transient behavior induced from vibration measured at the bearings, a nonlinear state observer is designed to detect cracks on the shaft. This is the elementary NL-observer(Beo). Using the observer, an Estimator(Observer Bank) is established and arranged at the certain position on the shaft. When a crack position is localized, the procedure for estimating of the depth is engaged.

Key Words : Dynamic Behavior, NL-Observer, Estimator, Crack Detection, Crack Position, Crack Depth

1. Introduction

As the classical method of a crack detection, there are some ways to find the split on the shaft. For example, some of them analyse the vibration peaks and acoustics and measure the oil temperature by the costdown and by the transition of the resonance (Muehlenfeld, 1992). Experts on these subject fail to find cracks very often and even the crack information is misunderstood as an effect from a damaged bearing. Except the analysis method, there are some other methods: namely, to compare the time signal between damaged stage in the operation and undamaged stage in the initial stage (Imam, 1987), to look for the sensibility of eigenvalue (Natke, 1990) and to use modal observer under model reduction. A similar way to detect a crack is also given by Soeffker (1993). But as a physical model they have used lumped-mass model, the results contain some error by the model reduction and physical modeling. These methods do not offer clear relationships between phenomena and change of the stiffness which are necessary for a crack detection on the shaft. There is not any method to localize the crack position on the shaft. Therefore, in this study a new method based on the theory of disturbance rejection control (Mueller, 1990; Mueller, 1993) is suggested for detecting crack and estimating the position with respect to constant crack depth. As an indicator for the existence of a crack, the nonlinear dynamic effects appeared by the change of the stiffness coefficients due to the rotation of the cracked shaft, are investigated. These effects related to the measurement on the bearings are important to determine the existence of the crack on the rotating shaft. But it is very difficult to set up the clear relation between crack and caused phenomena in the time domain operation. This is the main task in the area of the crack problem, too.

First of all, the basic state observer is estab-

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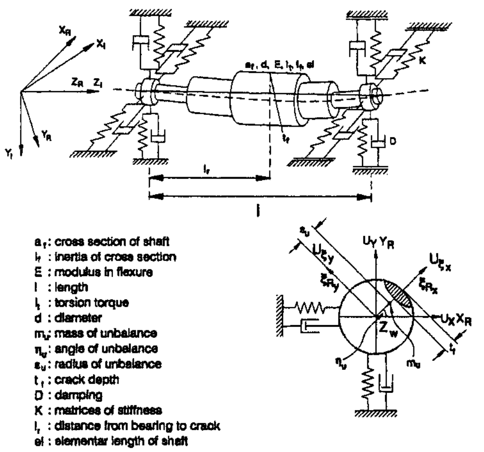

Fig. 1 Physical model of the rotor

lished in the way to modify the given system into the extended system with a linear fictitious model for the nonlinear system behavior. In this consideration, the effects of the extended system are interpreted as internal or external disturbance which is unknown at the initial stage.

The unknown nonlinear effects are approximated by the additional time signals of an by elementary state observer. FEM model, does not need calculate the relative compliance of the crack. Normally the elementary stiffness matrix for an undamaged rotor is given in the stage of the construction and the stiffness corresponding to the crack can be calculated (Waller, 1989; Link, 1989; Bathe, 1990).

As an example of the physical model, the shaft is modelled into N(=7) finite sub-shafts (Park, 1998); each one is called a subsystem. At both ends of the shaft there exist dynamics of the bearings. They have the task of system control. For the initial data needed in the operating system, the displacements of the jounals are measured on the bearings at the left and right sides of the shaft. It is assumed that the material properties are homogenous. The geometrical data and other detailed information are given in the appendix.

2. Equation of Motion

Assuming that there is only small deviation from motion and no redundant coordinate (Bremer, 1992; Bremer, 1988), the system including three harmonic unbalances in the 3rd, 4th and 5th subsystems in the middle of the shaft. Then the following equation can be accepted as linear system.

$$M_g \ddot{q}(t) + (D_{dg} + G_g)\dot{q}(t) + K_g q(t) = f_u(t) + f_g(t) + Ls(j)\ n(q(t), t) \quad (1)$$

Here, the index g denotes the whole system. Equation (1) is able to be discretized into N(=7) sub-finite systems and its equation of motion with crack at a subsystem j is described by

$$i_e = 1, \cdots, N \quad (2)$$

$$j_k(i_e) = \left[(i_e - 1)\frac{n}{2} + 1 \right]_{(i_e: 1, \cdots, N)} \quad (3)$$

$$i = j_k, \cdots, j_k + n - 1 \quad (4)$$

$$j = j_k, \cdots, j_k + n - 1 \quad (5)$$

With i_e, j_k, i and j the vector in explicit form and the equation of motion can be given as follows:

$$q_{(i_e+1)(i)(i=1,\cdots,\frac{n}{2}+1)} = q_{(i_e-1)(\frac{n}{2}+i)} \quad (6)$$

$$\sum_{i_e=1}^{N} \sum_{j_k=j_k(i_e)}^{j_k(i_e)+n-1} [M_e \ddot{q}_{j_k(i_e)}(t) + (D_e + G_e)\dot{q}_{j_k(i_e)}(t) + K_e q_{j_k(i_e)}(t)] = [f_u(t)](i_e)_{(i_e=3,4,5)} + [f_g(t)](i_e)_{(i_e=1,\cdots,N)} + Ls(n_f, i_e)[n(q_{(t)}(t), t)]_{(i_e=1,\cdots,N)} \quad (7)$$

where, the index e represents the elementary subsystem. The elementary notations in the equations denote as follows:

- $q(t)$, $\dot{q}(t)$, $\ddot{q}(t)$: displacement vector, velocity vector, and acceleration of the system
- M_g, K_g : mass matrix, stiffness matrix of undamaged section
- D_{dg}, $G_g = -G_g^T$: matrix of the damping and gyroscopic matrix
- $q_e(t)$, $\dot{q}_e(t)$, $\ddot{q}_e(t)$: displacement vector, velocity vector, and acceleration of the elementary subsystems. $q_e(t) \in \Re^n$, n(=8) and nn(=32) are degrees of freedom of considered elementary subsystem and total system. The $q_e(t)$ consists of $q_e(t) = (x_l, y_l, \theta_{xl}, \theta_{yl}; x_r, y_r, \theta_{xr}, \theta_{yr})$, the indices l and r denote the left and right nodes and x_r, y_r, θ_{xr}, θ_{yr} are the coordinates at the subsystem
- $f_u(t)$, $f_g(t)$, n(q(t), t) : vector of unbalance, gravitation input vector, and vector of the nonlinearities caused by unexpected influence (crack)

- M_e, K_e : mass matrix, stiffness matrix of undamaged secton
- D_{de}, $G_e = -G_e^T$, $Ls_{(n_f, i_e)}$: matrix of the damping, gyroscopic effects, and distribution vector with regard to the crack at subshaft number i_e

All system matrices are constant in terms of time t (Bremer, 1988; Bremer, 1992) and the distribution matrix (Park and Mueller, 1997; Park and Cho, 1998) is given in the following way:

$$Ls(i_e) = \begin{bmatrix} 000, \cdots, \overbrace{1000}^{i_e\text{-th position}}, \cdots, 000 \\ 000, \cdots, \underbrace{0100}_{i_e\text{-th position}}, \cdots, 000 \end{bmatrix}^T_{(2\times N)} \tag{8}$$

From now on, the index j will be left out with respect to the whole dynamic system. It is normally convenient for further operation to write the equation above via state space notation with $x(t) = [q(t)^T, \dot{q}(t)^T]$ including the nonlinearities of the motion created by a crack.

$$\dot{x}(t) = Ax(t) + Bu(t) + N_R n_R(x(t)) \tag{9}$$

The equation of the measurement is given by

$$y = Cx(t) \tag{10}$$

where, A is $(N_n \times N_n)$ dimensional system matrix which is responsible for the system dynamic with $N_n = 2nn$, u(t) denotes r-dimensional vector of the excitation inputs due to gravitation and unbalances and C presents $(m_e \times N_n)$-dimensional measurement matrix. W is the $(N_n \times N_n)$ dimensional matrix and $s(t)$ presents the plant vector of noise. w_m denotes the white measurement noise. x(t) is N_n-dimensional state vector, and y (t) is m_e-dimensional vector of measurements, respectively. Here, the vector $n_R(x(t))$ characterizes the n_f-dimensional vector of nonlinear functions due to the crack. N_R is the input matrix of the nonlinearties and the order of N_R is of $(Nn \times n_f)$. It is assumed that the matrices A, B, C, N_R, and the vector u(t), y(t) are already known. Where the weighting matrix Q corresponding to the plant and R_m regarding to the measurement should be suitably chosen by the trial and errors.

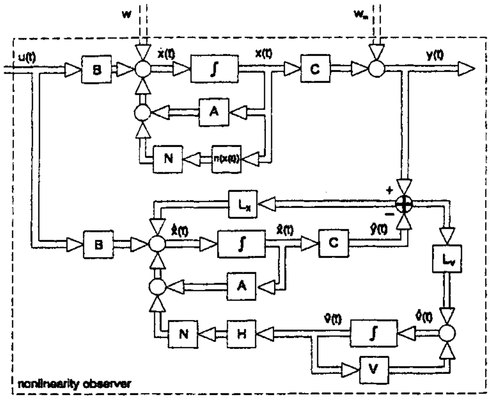

Fig. 2 Elementary observer (Beo)

Now it remains to reconstruct the unknown nonlinear vector $n_R(x(t), t)$ which mentions the disturbance force caused by a crack. The basic idea is to get the signals from $n_R(x(t))$ approximated by the linear fictitious model (Mueller, 1993)

$$n_R(x(t), t) \approx Hv(t) \tag{11}$$

$$\dot{v}(t) = Vv(t) \tag{12}$$

$$dim\ v(t) = s \tag{13}$$

The model describes the time behavior of the nonlinearities due to the appearance of the crack approximately as follows:

$$n_R(x(t), t) \approx \hat{n}_R(\hat{x}(t)) = H\hat{v}(t) \tag{14}$$

where $\hat{v}(t)$ follows from Eq. (18). The matrices H and V must be chosen according to the technical background considered in terms of oscillator model or integrator model (Mueller, 1990; Mueller, 1993). To obtain the signals $\hat{n}(\hat{x}(t))$ the elementary observer (Beo) should be designed.

At first, the given system of Eq. (9) must be extended with the fictitious model Eqs. (11) and (12) into extended model

$$\underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix}}_{\dot{x}_e(t)} = \underbrace{\begin{bmatrix} A & N_R H \\ 0 & V \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}}_{x_e(t)} + \begin{bmatrix} I \\ 0 \end{bmatrix} \tilde{u}(t) \tag{15}$$

$$y(t) = \underbrace{[C : 0]}_{C_e} \begin{bmatrix} x(t) \\ \cdots \\ v(t) \end{bmatrix} \tag{16}$$

where $N_R H$ couples the fictitious model of Eqs.

(11), (12) to the whole system. To enable the successful estimate, the number of the measurements must be at least equal or greater than the modelled nonlinearities $(m_e \geq n_f)$. When the above requirements are satisfied, the elementary observer in terms of an identity observer can be designed as follows:

$$\underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix}}_{\dot{x}_o(t)} = \underbrace{\begin{bmatrix} A-L_xC & N_RH \\ -L_vC & V \end{bmatrix}}_{A_o} \underbrace{\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}}_{x_o(t)} \underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix}}_{B_e} \tilde{u}(t) + \underbrace{\begin{bmatrix} L_x \\ L_v \end{bmatrix}}_{L_o} y(t) \quad (17)$$

$$\hat{y}(t) = \underbrace{[C : 0]}_{C_e} \begin{bmatrix} \hat{x}(t) \\ \cdots \\ \hat{v}(t) \end{bmatrix} \quad (18)$$

where matrices L_x and L_v are the gain matrix of the observer and white noise vector related to the state measurements, respectively. Equation (17) means that the observer consists of a simulated model with a correction feedback of the estimation error between real and simulated measurements. The matrix A_o has $(N_n+n_f \times N_n+n_f)$-dimensions and represents the dynamic behavior of the elementary observer. The asymptotic stability of the elementary observer can be guaranteed by a suitable design of the gain matrices L_x and L_v which are possible under the conditions of detectability or observabilty of the extended system of Eqs. (15), (16). For estimation under the asymptotic stability, the eigenvalue of the observer (A_o) must be settled on the left side of the eigenvalue of the given system(A_e) to make the dynamic of the observer faster than the dynamic of the system. The model for the crack behaviors can be designed using integrator model (Park and Mueller, 1997; Park and Cho, 1998) based on the choosen crack model (Bremer, 1992) as follows:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (19)$$

$$V = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (20)$$

$$n_{(R:1,\, x(t)_1)} \approx v_1(t) \quad (21)$$

$$n_{(R:2,\, x(t)_2)} \approx v_2(t) \quad (22)$$

The observer gain matrices L_x and L_v can be calculated by pole assignment or by the Riccati equation (Mueller, 1990; Mueller, 1993) as follows:

$$A+P+P\ A^T-P\ C^T R_m^{-1} C\ P+Q=0 \quad (23)$$

$$\begin{bmatrix} L_x \\ \cdots \\ L_v \end{bmatrix} = P\ C^T R_m^{-1} \quad (24)$$

The weighting matrix Q and R_m are suitably chosen by the trial and errors.

3. Design of an Estimator for the Localization

In the above section we studied how to design the elementary observer (Beo) for the detection at a given local position: a certain place on the shaft is initially given as the position. In the real running operation there is not any information about the position of the crack, so the elementary observer must survey not only the assigned local position, but also any other place on the shaft; it must give the signals whether a crack exists or not. Once it is known, it is possible to detect the crack on the shaft. When a crack appears at any subsystem in running time, it must be detected as well. But in many cases it has been shown that it is hard to estimate the position of the crack at all subsystems on the shaft with one Beo: it depends on the number of the subsystem and the number of Beo. For the estimation of a crack position we designed a method based on Estimator. The main idea is to feel the related crack forces from a certain local position to the arranged elementary observer.

Figure 3 shows the structure of the Estimator (Observer Bank). The number of elementary observers depends on the number of the subsystems modeled. Every elementary observer which is distinguished from the distribution matrix $L_{S(i_e)}$ has the same input (excitation) u(t) and the feedback of the measurements, and is set up at a suitable place on the given system. For the suitable arrangement of the Beo, the distribution matrix on the analogy of Eq. (8) is applied. To estimate the local place of the crack, there are two steps. First of all, the Beo must be observable to certain local in the meaning of the asymptotical stability in the system. The requirement has been satisfied by the criteria from Hautus (Mueller, 1990).

$$Rank\begin{bmatrix} \lambda I_{Nn}-A & -N_R(Ls_{(i)})H \\ 0 & \lambda I_{n_f}-V \\ C_e & 0 \end{bmatrix}=dim(x_e(t))$$
$$+dim(v(t))=N_n+n_f(=s) \qquad (25)$$

This means that the Beo must estimate the crack at any location, where Beo is situated on the given system.

The unknown crack position is to be found by the Beo arranged in a certain local place with the related crack forces resulting from the crack. To guarantee this, the condition of Eq. (25) must be fulfilled. In this work three Beos are arranged at the 2nd, 4th subsystems and the 6th like this:

$$Ls_{(2)}(i=2)=1,\ \text{otherwise } Ls_{(2)}(i)=0$$
$$Ls_{(4)}(i=15)=1,\ \text{otherwise } Ls_{(4)}(i)=0$$
$$Ls_{(6)}(i=30)=1,\ \text{otherwise } Ls_{(6)}(i)=0.$$

The equation of the estimator with the 1st Beo A at the 2nd subsystem, the 2nd Beo C at the 6th subsystem and the 3rd Beo B 4th at subsystem are described by

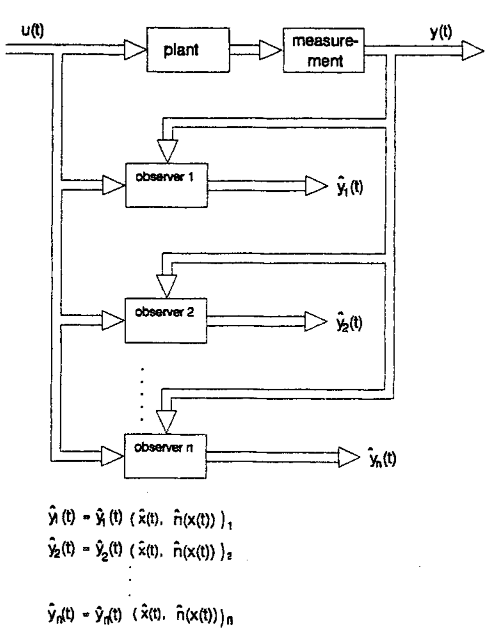

Fig. 3 Estimator (Observer Bank)

$$\begin{bmatrix}\dot{\hat{x}}(t)\\ \dot{\hat{v}}(t)\end{bmatrix}_2=\begin{bmatrix}A-L_{x_2}C & N_v(Ls(2))\ H\\ -L_{v_2}C & V\end{bmatrix}\begin{bmatrix}\hat{x}(t)\\ \hat{v}(t)\end{bmatrix}+\begin{bmatrix}b_r(t)\\ 0\end{bmatrix}+\begin{bmatrix}L_{x_2}\\ L_{v_2}\end{bmatrix}y(t)$$
$$\begin{bmatrix}\dot{\hat{x}}(t)\\ \dot{\hat{v}}(t)\end{bmatrix}_4=\begin{bmatrix}A-L_{x_4}C & N_v(Ls(4))\ H\\ -L_{v_4}C & V\end{bmatrix}\begin{bmatrix}\hat{x}(t)\\ \hat{v}(t)\end{bmatrix}+\begin{bmatrix}b_r(t)\\ 0\end{bmatrix}+\begin{bmatrix}L_{x_4}\\ L_{v_4}\end{bmatrix}y(t) \qquad (26)$$
$$\begin{bmatrix}\dot{\hat{x}}(t)\\ \dot{\hat{v}}(t)\end{bmatrix}_6=\begin{bmatrix}A-L_{x_6}C & N_v(Ls(6))\ H\\ -L_{v_6}C & V\end{bmatrix}\begin{bmatrix}\hat{x}(t)\\ \hat{v}(t)\end{bmatrix}+\begin{bmatrix}b_r(t)\\ 0\end{bmatrix}+\begin{bmatrix}L_{x_6}\\ L_{v_6}\end{bmatrix}y(t)$$

4. Examples

The Estimator consists of three Beo. The first Beo A is situated at the 2nd subsystem, the 2nd Beo B is at the 6th subsystem and the 3rd Beo C is placed at the 4th subsystem. The criterion to detect a crack is the magnitude of the crack forces. In order to localize a crack position, it is necessary to choose the maximal magnitude of the crack force from all Beo by the comparison among the forces turn out. In the case, the estimator shows none of the crack force; there is not any crack in this system considered. If any one of the Beo gives the signal of a force, the system has a crack in a corresponding position. As the 1st example, the given crack is at the 1st of the node in the system considered. Figure 4 shows that the estimator recognizes the appearance of a

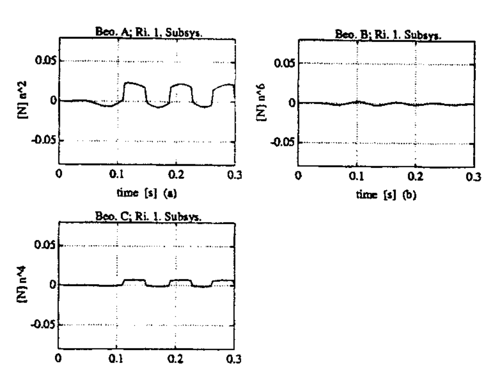

Fig. 4 Beo A, B, C: Crack in the 1st Subsystem, $t_{(ri:1)}=0.135$, $t_{(s)}=0.03[s]$, Y coordinate: crack force in N, X coordinate: time in sec.

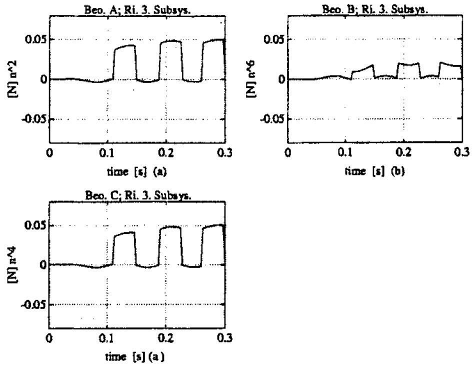

Fig. 5 Beo, A, B, C: Crack in the 3rd Subsystem, $t_{(ri)}=0.15$, $t_{(s)}=0.03[s]$, Y coordinate: crack force in N, X coordinate: time in sec.

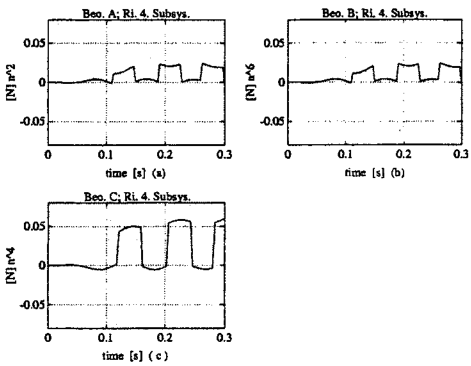

Fig. 6 Beo, A, B, C: Crack in the 4th Subsystem, $t_{(ri)}=0.15$, $t_{(s)}=0.03[s]$, Y coordinate: crack force in N, X coordinate: time in sec.

crack at the time 0.135. By the comparison of the forces, the elementary observer Beo A sees the lagest crack force. It means that the crack is closer or near to the 1st of the node (Beo A) than to the 6th of the node(Beo B) and the 4th of the node (Beo C). Although the Beo B is hardly to know the existence of a crack, the detection and localization of the crack is succesful.

As the 2nd example, the given crack is situated at the 3rd of the node in the system considered.

In Fig. 5 the elementary observer Beo A and Beo C see the same magnitude of the crack forces. It denotes that the signals have been turned out between Beo A and Beo C. The estimator estimates the crack position between the 2nd node and the 4th node in the system. It is the 3rd node.

As the 3rd example, the given crack is at the 4th

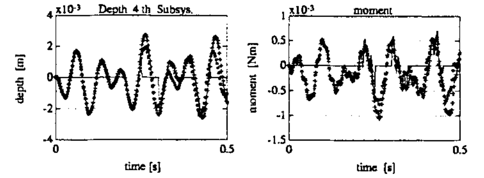

Fig. 7 Crack depth in the 4th of the node with 25 db ratio in measurement, $t_{(f)}=0.002m$; $t_{(ri)}=0.2$ $[s]$, x coordinate: time in $[s]$, y coodinate depth in $[m]$

of the node in the system considered.

Figure 6 illustrates the lagest magnitude of the crack force by the elementary observer Beo C. It tells that the crack is placed at the 4th node. Like the 1st and the 2nd examples, the estimator gives the information where a crack appeared. In this way the Estimator estimates the existence of a crack by the crack force and localize its position according to the magnitude of a crack forces. These forces related from certain position of a crack to Beo A, Beo B and Beo C are interpreted as mechanical forces due to the breathing and gaping from Gasch model (Gasch, 1976). The numerical value of the ρ_q concerned with the weighting matrix Q is in the appendex. The factor ρ_r of the weighting matrix R_m is 0.975 and *diag* $R_{(i,j)}$ is 1. The matrices Q and R_m are chosen by the trial and errors. The external signal exists in case of the opened crack. On analogy of the system model, the minimal and maximal values depend on the depth. If only the crack is situated at the position where the Beo is located. Otherwise the position of the crack plays a part in the values of the forces regarding to the excited inputs as well. However, the crack forces are a clear indicator for the appearance of a crack in operating time. The other results which are ouitted this paper, show that Beo B arranged at the right bearing, can not estimate the crack in the 1st of the node in the system. In the simulation the given depth is 2 mm and the time of appearance of the crack is 0.2 sec.

As an estimation of depth, the displacement of the vertical direction is taken. The depth given is 2mm at the 4th of the node.

Plots of Fig. 7 show the crack depth and the moment at the middle position as an example.

The result shows that the displacement is almost not corresponding to given depth. The maximal magnitude of the depth estimated is of 3mm. But this is an approximate way to estimate crack depth. This is a remained task to be researched in the future.

5. Summary and Conclusions

Using the FEM, the mathematical model of the rotating shaft including a crack has been presented. Based on the mathematical model, the elementary observer and an estimator have been developed. With this estimator, the task of the crack detection and localization have been made. The above methods give a clear relation between the damaged shaft by a crack and the caused phenomena in vibration by means of the measurement at both bearings. Theoretical results have been given. The forces in the results are the internal forces which have been reconstructed as disturbance forces created by the crack.

From the given examples, it has been theoretically shown that the cracks on the shaft can be detected. The Estimator is able to estimate the location of a crack. The method considered can be applied to the similar area with the nonlinear dynamic effect from a crack problem by the suitable design of an Estimator. The suggested methods are very significant not only for the further theoretical research and developments but also for the transfer in the experiments.

For the estimation of depth it is needed to establish the relationship between crack force and the corresponding displacement. This is a remained task to be researched furthermore.

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Appendix

Using the abbreviation $ii=i-j_k+1$, $jj=j-j_k+1$, the sum of the matrices, with accordance to Eqs. (2) and (3), can be described as follows.

$$M_{(g)(j_k,j_k)}(i_e)=\sum_{i_e=1}^{N}\left[\sum_{j_k=1}^{(i_e-1)\frac{n}{2}+1}\left(\sum_{i,j=j_k}^{j_k+n-1}M_e(ii,\ jj)\right.\right.$$
$$\left.\left.+M^0_{(dime\times dime)}\right.\right. \qquad \text{(A1)}$$

$$K_{(g)(j_k,j_k)}(i_e)=\sum_{ie=1}^{N}\left[\sum_{j_k=1}^{(i_e-1)\frac{n}{2}+1}\left(\sum_{i,j=j_k}^{j_k+n-1}K_e(ii,jj)\right.\right.$$
$$+K^0_{(dime\times dime)} \quad (A2)$$

$$G_{(g)(j_k,j_k)}(i_e)=\sum_{ie=1}^{N}\left[\sum_{j_k=1}^{(i_e-1)\frac{n}{2}+1}\left(\sum_{i,j=j_k}^{j_k+n-1}G_e(ii,jj)\right.\right.$$
$$+G^0_{(dime\times dime)} \quad (A3)$$

$$D_{(g)(j_k,j_k)}(i_e)=\sum_{ie=1}^{N}\left[\sum_{j_k=1}^{(i_e-1)\frac{n}{2}+1}\left(\sum_{i,j=j_k}^{j_k+n-1}D_e(ii,jj)\right.\right.$$
$$+D^0_{(dime\times dime)} \quad (A4)$$

The matrices used in Eq. (9) are follows

$$A=\begin{bmatrix} 0 & \vdots & I_{(nn)} \\ \cdots & \cdots & \cdots \\ -(M_g)^{-1}K_e & \vdots & -(M_g)^{-1}(D_{dg}+G_e) \end{bmatrix}_{(64\times 64)} \quad (A5)$$

The index i denotes the number of the subsystem. The vector of the order of the excitation and the matrix of nonlinearites,

$$\tilde{u}(t)=\begin{bmatrix} 0 \\ \cdots \\ M_g^{-1}\ f_e \end{bmatrix}_{(64\times 1)} \quad (A6)$$

$$N_R(L_{S(i)})=\begin{bmatrix} 0 \\ \cdots \\ -M_g^{-1}L_{S(i)} \end{bmatrix}_{(64\times 1)} \quad (A7)$$

is of (64×1).

where the vector of the excitation consists of graviation and harmonic unbalance, is presented by

$$\begin{cases} f_e=f_{(g,\ i_e;\ i=1,\cdots,N)}+f_{(u,\ i_e=3,4,5)} \\ f_{(g;\ 2)}=f_{(g;\ 30)}=0 \\ ,\ f_{(g;\ 6)}=f_{(g;\ 10)}=f_{(g;\ 14)}= \\ f_{(g;\ 18)}=f_{(g;\ 22)}=f_{(g;\ 26)}=-mg, \end{cases} \quad (A8)$$

The order of the f_g is of (32×1) and f_u is of (32×1).

$$\begin{cases} {}_{(u;\ 17)}=f_{(u;\ 21)}=f_{(u;\ 25)}= \\ -e_m\Omega^2 m_{(ex)}\ sin(\Omega t+\beta) \\ f_{(u;\ 18)}=f_{(u;\ 22)}=f_{(u;\ 26)}= \\ e_m\Omega^2 m_{(ex)} cos(\Omega t+\beta) \end{cases} \quad (A9)$$

where angle of the phase: $\beta=0$, length of the subsystem of rotor $el=2m$, Diameter of the subsystem of rotor makes $ed=0.25m$. The mass of elemental subsystem: $m=\pi\ el\ \rho\frac{ED^2}{4}$, The density is of $\rho=7860\ \frac{kg}{m^3}$ excentricity: $e_m=0.0001$, mass of the excentricity: $m_{(ex)}=3$ m respectively. The modulus $E\ i_f$ is of $2.0\times 10^5 N/mm^2$. The stiffness of bearing: $K_{beaing}=15\times 10^5 N/mm^2$. The measurement matrix of order(4×64), $C_{(i=1,\cdots,j=1,\cdots,64)}=0$, except $C_{(1\times 1)}=C_{(2\times 2)}=C_{(29\times 29)}=C_{(30\times 30)}=1$. The number of the nonlinearities n_f are of 1 and the number of the measurements m_e makes 4. The elementar matrices K_e, M_e and D_g which depend on the geometry, are given in (Waller and schmidt, 1989; Link, 1989; Barthe, 1990). The weighting matrix $Q_{q2}(i=1,\cdots,66,j=1,\cdots,66)$ and $Q_{q6}(i=1,\cdots,66,j=1,\cdots,66)$ is of:

$$\begin{cases} Q(i,j)=6\cdot 10^5;\ i=j=1,\cdots,16 \\ Q(i,j)=5\cdot 10^5;\ i=j=17,\cdots,32 \\ Q(i,j)=15\cdot 10^5,\ i=j=33,\cdots,45 \\ Q(i,j)=2.5\cdot 10^5,\ i=j=46,\cdots,64 \\ Q(i,j)=5\cdot 10^7;\ i=j=65,\cdots,66. \end{cases} \quad (A10)$$